

# Langlands parameters of symmetric unitary matrix models

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## Abstract

It follows from work of Anagnostopoulos-Bowick-Schwarz that the partition function of the symmetric unitary matrix model can be described via a certain pair of points in the big cell of the Sato Grassmannian. We use their work to attach a connection on the formal punctured disc – hence a local geometric Langlands parameter – to the matrix model which governs the theory. We determine the Levelt-Turittin normal form explicitly in terms of the coefficients of the potential and relate the classical limit with the spectral curve of the matrix model. In contrast to D-modules attached by Schwarz and Dijkgraaf-Hollands-Sulkowski to the Hermitian matrix model, in the unitary case the connection turns out to be reducible. We embed our discussion into a more general analysis of D-modules attached to quivers in the Sato Grassmannian, we clarify some inaccuracies in the literature concerning Virasoro constraints of unitary matrix models, and we extend the classification of solutions to the string equation of the unitary matrix model to much more general quivers.

## 1 Introduction

Dijkgraaf in [6], and later in collaboration with Hollands, Sulkowski and Vafa [7], [8], emphasized the usefulness of D-modules in understanding and describing partition functions of quantum field theories. Notable examples treated by these methods are 2D quantum gravity, Seiberg-Witten theory, and  $c = 1$  string theory.

The starting point of the present work is to extend the D-module techniques to quantum field theories related to unitary matrix models, such as 2D quantum chromodynamics with gauge group a unitary group. The second aim of our work is to initiate a somewhat general framework that encompasses several of the above examples. We do this by studying quivers in the Sato Grassmannian, see Section 2 for definitions, and by attaching suitable D-modules to them.

To understand the origins of these D-modules one should note the following: In the case of 2D gravity, Dijkgraaf-Hollands-Sulkowski in [7] were only able to find D-modules for the generalized topological models, meaning the  $(p, 1)$  models. An important advance was made in the work of Schwarz [18]: He found D-modules, the so-called companion matrix connections, for the general  $(p, q)$  models. From another perspective, these D-modules on the formal punctured disc describe the Kac-Schwarz operators.

This brings us to the important point that for general quivers the companion matrix D-module need not agree with the Kac-Schwarz D-module: the latter is not even always possible to define. It turns out that the two notions happen to agree in the case of  $(p, q)$  minimal models coupled to gravity. We elucidate their Levelt-Turittin normal form in Theorems 1, 2, and 3. We apply these results to give generalizations of the classification by Anagnostopoulos-Bowick-Schwarz [2] of the solutions to the string equation of the

symmetric unitary matrix model.

The underlying space of the D-modules that we consider is the formal punctured disc, hence a D-module is the same as a connection. With the additional data of an oper structure, the local geometric Langlands correspondence of Frenkel-Gaitsgory [10] attaches to such objects a categorical representation of the affine Lie algebra  $\widehat{\mathfrak{gl}}_n$  at the critical level, where  $n$  is the dimension of the connection in question. This correspondence is supposed to be independent of the choice of oper structure. One can say that we attach local geometric Langlands parameters to a wide class of quivers in the Sato Grassmannian, including the case of interest in the Hermitian and unitary matrix models.

This Langlands theoretic viewpoint has proven useful already: Namely, in [15], [17] previously used symmetries of corresponding arithmetic local Langlands parameters were transferred to the geometric setting to give a proof of the T-duality of minimal conformal matter coupled to gravity. See [16] for the relation between the T-duality and the local arithmetic Langlands duality.

The current paper is structured in the following manner: In Section 2 we define the various notions of quivers in the Sato Grassmannian that we are interested in. In Section 3 we attach two kinds of D-modules to the quivers and obtain results concerning their normal forms and use them to obtain a classification result of solutions to quivers generalizing [2]. In Section 4 we describe the Virasoro constraints of the  $\tau$ -functions attached to quivers and we clarify an apparent gap in the literature concerning Virasoro constraints of unitary matrix models.

On a future occasion we plan to describe the aspects of our constructions related to the representation theory of affine Lie algebras in more detail and clarify for example natural oper structures on the D-modules occurring in the present work.

## 2 Quivers in the Sato Grassmannian

Consider a quantum field theory partition function  $Z$ . The Dyson-Schwinger equations put constraints on  $Z$  but it can be difficult to classify the possible solutions to the system of equations. The situation simplifies considerably if  $Z$  can be related to  $\tau$ -functions of integrable systems. One can then ask what further constraints in addition to the integrability one has to impose in order to describe the partition function. In favorable circumstances a finite number of additional constraints suffices:

An important example is the treatment by Kac-Schwarz [14] of  $(p, q)$  minimal conformal matter couple to gravity via what is now known as Kac-Schwarz operators. The key is to find a point of the big cell of the Sato Grassmannian stabilized by certain two operators. This example is related to double scaling limits of Hermitian matrix models. It turns out that for double scaled symmetric unitary matrix models (UMM) the corresponding point configuration in the Sato Grassmannian consists of two points with certain operators mapping between the two.

These two examples naturally lead to the study of what we call quivers in the Sato Grassmannian. See the work of Adler-Morozov-Shiota-van Moerbeke [1] for another generalization of the Kac-Schwarz operators. To motivate our definitions we describe, following [2], the special case of Grassmannian quiver relevant for the UMM.

**Definition 1.** Fix an indeterminate  $z$  and let  $Gr$  denote the big cell of the index zero part of the Sato Grassmannian: It consists of complex subspaces of  $\mathbb{C}((1/z))$  whose projection to  $\mathbb{C}[z]$  is an isomorphism.

It is known, see for example the work of Anagnostopoulos-Bowick-Schwarz [2], that

the UMM partition function satisfies

$$Z = \tau_1 \cdot \tau_2$$

where  $\tau_1$  and  $\tau_2$  are KdV  $\tau$ -functions whose associated points  $V_1$  and  $V_2$  of the Sato Grassmannian satisfy the constraints

- (i)  $zV_1 \subset V_2$
- (ii)  $zV_2 \subset V_1$
- (iii)  $AV_1 \subset V_2$
- (iv)  $AV_2 \subset V_1$

where  $A$  is an operator depending on the potential of the matrix model, its simplest incarnation is of the form

$$A = \frac{d}{dz} + z^{2n}$$

for some  $n \geq 1$ . The pair  $(V_1, V_2)$  together with the conditions on the  $z$  and  $A$  action are a special case of what we call a quiver in the Sato Grassmannian:

**Definition 2.** Fix  $n \geq 1$ . A quiver denotes a collection of operators

$$U_k \in \mathbb{C}[z, \partial_z], \quad k \geq 1$$

together with the data for each  $k$  of a subset  $T_k \subseteq \{1, 2, \dots, n\}$  and an injective map

$$s_k : T_k \rightarrow \{1, 2, \dots, n\}$$

A solution to the quiver is defined to be an  $n$ -tuple  $(V_1, \dots, V_n)$  of points of the big cell of the Sato Grassmannian such that

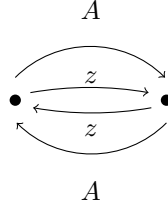
$$U_k V_i \subseteq V_{s_k(i)}$$

for all  $i \in T_k$ . We call  $n$  the degree of the quiver.

It would be more accurate to call this a quiver representation, but for brevity of notation we simply call it a quiver.

Note that in all the examples that we consider one has  $1 \leq k \leq 2$ . Furthermore, a solution to a quiver is simply an  $n$ -tuple of points in the Grassmannian which are related by suitable operators in a prescribed manner. We often describe such quivers and their solutions by letting the  $V_i$ 's correspond to vertices of a graph, and the operators  $U_k$  via a collection of directed edges.

For example, the UMM partition function can be described via the following quiver:



We sketch how Anagnostopoulos-Bowick-Schwarz in [2] arrive at this description:

Fix a positive integer  $N$  and a polynomial potential

$$V(X) = \sum_{i \geq 0} t_i X^i$$

The partition function of the symmetric unitary one-matrix model is defined as

$$Z_N(t_1, t_2, \dots) = \int_{U(n)} \exp\left(-\frac{N \cdot \text{Tr } V(X + X^\dagger)}{\lambda}\right) d\mu$$

where  $d\mu$  is a Haar measure on the group  $U(n)$  of  $n \times n$  unitary matrices and  $N = [n/2]$  and  $(-)^\dagger$  denotes the conjugate transpose. Let  $Z$  denote the partition function of the double scaling limit. The function  $v$  such that

$$v^2 \sim -\partial^2 \ln Z$$

is a solution of the modified KdV (mKdV) equation

$$v_t + v_{xxx} \pm 6v^2 v_x = 0$$

For a description of  $Z$  in terms of the Sato Grassmannian it is useful to relate  $v$  via the Miura transform to the usual KdV equation. The Miura transformation  $u_\pm = v^2 \pm v_x$  takes a solution  $v$  of the mKdV equation to a solution  $u_\pm$  of the KdV equation.

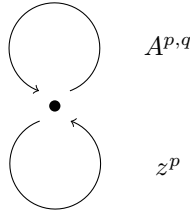
$$u_t + u_{xxx} + 6uu_x = 0$$

This can be generalized to the whole corresponding hierarchies of differential equations. Then

$$u_\pm = -2\partial^2 \ln \tau_\pm$$

where  $\tau_\pm$  is a  $\tau$ -function of the KdV hierarchy. The two points of the Sato Grassmannian corresponding to these two  $\tau$ -functions are then precisely the two vertices of the desired quiver that we described previously. The operators  $A$  and  $z$  are obtained from the string equation of the UMM, we refer to [2] for details.

Another important example of quivers related to quantum field theory concerns the  $(p, q)$  minimal model conformal field theory coupled to gravity. The corresponding partition function can be described as a square of a special KP  $\tau$ -function satisfying two additional constraints. In terms of the Sato Grassmannian the problem reduces to finding a point  $V \in Gr$  stabilized by  $z^p$  and the Kac-Schwarz operator  $A^{p,q}$ . Hence the situation is described by a quiver of the form

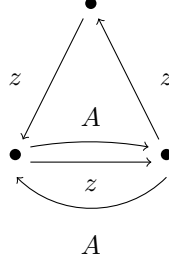


where

$$A^{p,q} = \frac{1}{pz^{p-1}} \frac{d}{dz} + \frac{1-p}{2p} \frac{1}{z^p} + z^q$$

The following slight deformation of the  $z$  action of the UMM quiver is a different kind

of example:



For example, when the top and lower left point collapse one has a special case of the UMM which can of course be solved easily directly:

One sees that in fact both points are the same and have to satisfy  $AV \subseteq V$ . If one writes

$$V = \text{span}_{\mathbb{C}}(\phi, z\phi, z^2\phi, \dots)$$

then the  $A$  constraint is equivalent to  $\partial_z \phi \in V$ . Since this derivative projects to 0 in  $\mathbb{C}[z]$  it follows that  $\phi$  has to be a constant and  $V$  is the vacuum point  $\mathcal{H}^+ = \mathbb{C}[z]$ .

In general, it is an interesting question to analyze for example how Virasoro constraints vary after slight variation of the form of the quiver.

We now define various special types of quivers that are the focus of the present work.

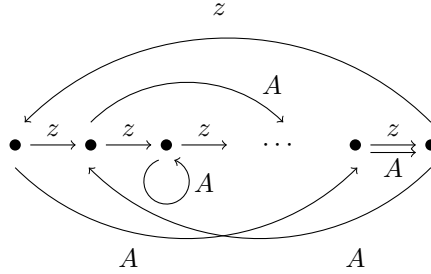
**Definition 3.** Fix  $p \geq 1$ . The  $z$ -cyclic quiver consists of  $(V_1, \dots, V_n)$  with each  $V_i$  in  $Gr$  such that

$$z^p V_i \subseteq V_{i+1}$$

where  $i + 1$  is taken modulo  $n$ .

The  $n = 2$  case (and  $p = 1$ ) correspond to the modified KdV hierarchy and is the case relevant for the UMM. The case of general  $n$ , again with  $p = 1$ , corresponds to the  $n$ -reduction of the modified KP hierarchy, see for example [5]. Note that usually the modified KP hierarchy is described via certain flags of points in the Sato Grassmannian not all of which lie in the big cell. Our formulation, just as in [2], corresponds to this description by scaling with suitable powers of  $z$ .

We are interested in the  $z$ -cyclic quiver together with a Kac-Schwarz operator  $A$ . Many of the UMM results can be generalized to quivers of the shape such as



A suitable class of such quivers is the following:

**Definition 4.** Fix  $p \geq 1$ ,  $f \in \mathbb{C}[z, z^{-1}]$ , and an  $n$ -cycle  $\sigma \in S_n$ . A permutation quiver is a  $z$ -cyclic quiver  $(V_1, \dots, V_n)$  such that

$$AV_i = \left( \frac{1}{pz^{p-1}} \partial_z + f(z) \right) V_i \subseteq V_{\sigma(i)}$$

for all  $1 \leq i \leq n$ . We denote this quiver by  $\text{Quiv}_f(\sigma)$ .

There is an important subclass of permutation quivers: Namely when one obtains a string equation from the  $z$  and  $A$ -action:

**Definition 5.** Fix  $p \geq 1$  and  $f \in \mathbb{C}[z, z^{-1}]$ . A string quiver is a permutation quiver such that the permutation  $\sigma$  is the inverse permutation of the one coming from the  $z$ -action:

$$\sigma = (n \ n-1 \ \dots \ 1)$$

For the rest of this work we mostly focus on the case  $p = 1$ , this case is sufficient for interesting generalizations of the UMM quiver. The case  $p > 1$  is needed when discussing generalizations of the quivers related to  $(p, q)$  minimal models coupled to gravity.

### 3 D-modules for quivers

For the  $(p, q)$  models of 2D quantum gravity the D-module approach of Dijkgraaf-Hollands-Sulkowski [7] and Schwarz [18] has proven useful, for example in proving the  $p - q$  duality. We now extend the D-module approach to permutation quivers.

An important point is that for the  $(p, q)$  models there are two D-modules on the punctured disc that turn out to be isomorphic: The first could be called the Kac-Schwarz (KS) connection and is based on interpreting the Kac-Schwarz operator of the  $(p, q)$  model as a  $p$ -dimensional connection. The second could be called, following [19], the companion matrix connection and is obtained from describing the action of the Kac-Schwarz operator on the relevant point of the Sato Grassmannian.

It turns out that in the generality of permutation quivers, the two notations do not agree in general. In fact, the KS connection can only be defined if the permutation of  $z$  and the Kac-Schwarz operator are inverses, as is the case for example in the UMM quiver. However, the companion matrix connection can always be defined.

#### 3.1 KS connection

The KS connection will turn out to be a connection on the formal punctured disc. Such an object is defined in the following manner:

**Definition 6.** A connection on the formal punctured disc  $\text{Spec } \mathbb{C}((t))$  is a (finite dimensional)  $\mathbb{C}((t))$ -vector space  $V$  together with a  $\mathbb{C}$ -linear endomorphism  $\nabla$  of  $V$  such that for all  $f \in \mathbb{C}((t))$  and  $v \in V$  one has  $\nabla(fv) = (\partial_t f)v + f\nabla(v)$ .

Suppose  $x = 1/t$  and  $M$  is a (finite rank) D-module on  $\mathbb{A}^1 = \text{Spec } \mathbb{C}[x]$ . Then the restriction to a punctured disc around  $\infty$  is defined as

$$\text{Res}_\infty(M) := M \otimes_{\mathbb{C}[x]} \mathbb{C}((t))$$

It naturally has the structure of connection on the formal puncture disc  $\text{Spec } \mathbb{C}((t))$ . The KS connection will be obtained via this restriction process.

In order to motivate our definition of KS connection  $\nabla_{\text{KS}}$ , see Definition 7, we explain how this concept naturally arises for the  $(p, q)$  minimal models coupled to gravity and also for the UMM. To put it succinctly, starting with a solution to the string equation  $[P, Q] = 1$  for suitable differential operators  $P$  and  $Q$ , one obtains a representation of the one-variable Weyl algebra  $\mathbb{C}[x, \partial_x]$  and hence a D-module on  $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1 = \text{Spec } \mathbb{C}[x]$ .

Then  $\nabla_{\text{KS}}$  is the restriction  $\text{Res}_\infty$  of this D-module to the formal punctured disc around  $\infty$ . This gives a connection on the formal punctured disc.

For the  $(p, q)$  models, the operators  $P$  and  $Q$  are in  $\mathbb{C}[[x]][\partial_x]$  and of degree  $p$  and  $q$ . Furthermore  $P$  is normalized, meaning it is monic with vanishing subleading term. There is a second viewpoint on these operators that will be useful for our later considerations. Since  $P$  is normalized it is known that there exists a monic degree 0 pseudodifferential operator  $S$  such that:

$$\begin{aligned} \text{(i)} \quad \tilde{P} &:= SPS^{-1} = \partial_x^p \\ \text{(ii)} \quad \tilde{Q} &:= SQS^{-1} = \partial_x^{1-p}x - \frac{1-p}{2p}\partial_x^{-p} - \sum_{-p < i \leq q} a_i \partial_x^i \end{aligned}$$

As usual,  $\partial_x$  acts on  $\mathbb{C}((1/z))$  by multiplication by  $z$  and  $x$  acts as  $-\partial_z$ . Then  $\tilde{P}$  acts on  $V := S\mathbb{C}[z]$  via multiplication by  $z^p$  and  $\tilde{Q}$  acts via the so-called Kac-Schwarz operator

$$A^{p,q} = \frac{1}{pz^{p-1}} \frac{d}{dz} + \frac{1-p}{2pz^p} + \sum_{-p < i \leq q} a_i z^i$$

Then the point  $V$  of the big cell has the structure of a D-module via the string equation

$$[\tilde{P}, \tilde{Q}] = [A^{p,q}, z^p] = 1$$

Let us now come to the analogous situation for the unitary matrix model. The operators  $P$  and  $Q$  are now certain  $2 \times 2$  matrix valued differential operators. In fact, there exist monic degree zero pseudodifferential operators  $S_1$  and  $S_2$  such that

$$\tilde{Q} := \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} Q \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix}^{-1} = \begin{bmatrix} & \partial_x \\ \partial_x & \end{bmatrix}$$

and

$$\tilde{P} := \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} P \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix}^{-1} = (-x + \sum_i a_i \partial_x^i) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

Let  $A$  be the Kac-Schwarz operator, meaning the element in  $\mathbb{C}[z, \partial_z]$  describing the action of  $\tilde{P}$  on  $\mathbb{C}((1/z))$ . The points  $V_1 := S_1\mathbb{C}[z]$  and  $V_2 := S_2\mathbb{C}[z]$  of the big cell of the Sato Grassmannian are the vertices of the desired quiver and it is clear that  $A$  maps  $V_1$  to  $V_2$  and  $V_2$  to  $V_1$ . One then obtains via the string equation  $[\tilde{P}, \tilde{Q}] = 1$  a D-module structures on  $S_1\mathbb{C}[z] \oplus S_2\mathbb{C}[z]$  via letting  $\partial_x$  act via  $-\tilde{Q}$  and  $x$  via  $\tilde{P}$ . Note that we choose this ordering, rather than letting  $\partial_x$  act via  $\tilde{Q}$  and  $x$  via  $\tilde{P}$ , in order to conform with certain conventions in the case of the string equation of Hermitian matrix models. After Theorem 1 we discuss the Fourier dual conventions.

### 3.1.1 String quivers

We now generalize the previous discussion by defining a KS connection for general string quivers. As usual, we assume  $p = 1$ . Let

$$Gr^n := Gr \times \cdots \times Gr$$

denote the  $n$ -fold product of the big cell of the Sato Grassmannian. A solution  $(V_1, \dots, V_n)$  to a string quiver will be of the form

$$(\gamma_1\mathbb{C}[z], \dots, \gamma_n\mathbb{C}[z]) \in Gr^n$$

for suitable monic degree 0 pseudo-differential operators  $\gamma_1, \dots, \gamma_n$ . Since for string quivers  $z$  and  $A$  act via inverse permutations and since  $[A, z] = 1$ , the  $\mathbb{C}$ -vector space

$$M := V_1 \oplus \dots \oplus V_n$$

carries a representations of the Weyl algebra by letting  $\partial_x$  act via  $A$  and  $x$  via  $z$ .

**Theorem 1.** *Fix a degree  $n$  string quiver  $\text{Quiv}_f(\sigma)$ . For every solution of the quiver, the module  $M$  is a free rank  $n$   $\mathbb{C}[x]$ -module and the isomorphism class of  $\nabla_{\text{KS}} := \text{Res}_\infty(M)$  is independent of the solution:*

$$\nabla_{\text{KS}} \cong \bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z))$$

where  $\zeta_n$  is a primitive  $n$ 'th root of unity.

*Proof.* One sees that  $M$  is a free rank  $n$   $\mathbb{C}[x]$ -module with basis

$$\{(\gamma_1 1, 0, \dots, 0), \dots, (0, \dots, 0, \gamma_i 1, 0, \dots, 0), \dots, (0, \dots, 0, \gamma_n 1)\}$$

It follows that  $\nabla_{\text{KS}}$  is an  $n$ -dimensional connection.

Now note that the eigenvalues of the relevant permutation matrix in the definition of  $\nabla_{\text{KS}}$  are  $\zeta_n^i$  for  $0 \leq i \leq n-1$ . Hence, after using a gauge transformation corresponding to the eigenvectors one obtains that

$$\begin{aligned} \partial_x &\mapsto (\partial_z + f(z)) \begin{bmatrix} 1 & & & \\ & \zeta_n & & \\ & & \ddots & \\ & & & \zeta_n^{n-1} \end{bmatrix} \\ x &\mapsto z \begin{bmatrix} 1 & & & \\ & \zeta_n^{-1} & & \\ & & \ddots & \\ & & & \zeta_n^{1-n} \end{bmatrix} \end{aligned}$$

Consider the  $i$ 'th element of the diagonal and let  $w = \zeta_n^{-i} z$ . Then

$$\zeta_n^i (\partial_z + f(z)) = \partial_w + \zeta_n^i f(\zeta_n^i z)$$

and hence the isomorphism class of  $\nabla_{\text{KS}}$  is as described in the theorem.  $\square$

We hence make the following definition:

**Definition 7.** The KS connection of a string quiver  $\text{Quiv}_f(\sigma)$  is defined to be  $\nabla_{\text{KS}}$ .

A remark about coordinate conventions:

Since we modeled the Sato Grassmannian on subspaces of  $\mathbb{C}((1/z))$  rather than  $\mathbb{C}((z))$  (both conventions are widely used), we usually write connections with respect to the variable  $z$ . So for example,  $(\mathbb{C}((z)), \partial_z + 1)$  denotes the 1-dimensional irregular connection usually denoted by  $(\mathbb{C}((\xi)), \partial_\xi - 1/\xi^2)$ .

The set of connections on the punctured disc forms a category when morphisms between two connection  $(V_1, \nabla_1)$  and  $(V_2, \nabla_2)$  are  $\mathbb{C}$ -linear maps  $g : V_1 \rightarrow V_2$  such that

$$g \circ \nabla_1 = \nabla_2 \circ g$$



One should then compare what the notion of gauge equivalence of connections, meaning isomorphism of connections, corresponds to on the level of the Sato Grassmannian for the above quivers.

Let

$$\Gamma_0 := \{1 + a_{-1}\partial^{-1} + a_{-2}\partial^{-2} + \dots \mid a_i \in \mathbb{C} \text{ for all } i\}$$

denote the group of constant coefficient monic degree zero pseudo-differential operators. These operators are often called the gauge transformations of the Sato Grassmannian. As indicated earlier, these are used in [2] to gauge away the negative powers of  $f$ , which in loc. cit. is a polynomial rather than a Laurent polynomial.

If one has a connection  $\nabla$  whose leading order coefficient matrix is diagonalizable then after diagonalizing that matrix the full connection can be diagonalized via a gauge transformation of the form  $\gamma = \mathbf{1}_n + \mathcal{O}(t) \in (\mathfrak{gl}_n \mathbb{C})[[t]]$ :

$$\nabla \sim \partial_t - \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{bmatrix}$$

for some  $f_i \in \mathbb{C}((t))$ . But for the diagonalized connection one can use gauge transformation of diagonal form to subtract from all the  $f_i$ 's all terms of order at most  $1/t^2$  and integer multiples of the regular terms of order  $1/t$ . This is exactly the role that the Sato Grassmannian gauge transformations have for each of the  $i$  points of the quiver. In this sense one sees the correspondence of gauge transformations.

Now note that the D-module  $\hat{M}$  that is globally Fourier dual to  $M$  is defined by letting  $\partial_x$  acts via  $-z$  and  $x$  via  $A$ . Let

$$\hat{\nabla}_{\text{KS}} := \text{Res}_\infty \hat{M}$$

From Theorem 1 one can deduce the gauge equivalence class of  $\hat{\nabla}_{\text{KS}}$  from  $\nabla_{\text{KS}}$ . Similar considerations were used in [15], [17] in the context of Hermitian matrix models to give a new proof of the p – q duality of 2D quantum gravity. This gives ample motivation to carry out the analogous calculations in the unitary case:

It follows from [4] (Proposition 3.12) that the Bloch-Esnault-Lopez local Fourier transform functor  $\mathcal{F}^{(\infty, \infty)}$  satisfies

$$\mathcal{F}^{(\infty, \infty)} \nabla_{\text{KS}} \cong \hat{\nabla}_{\text{KS}}$$

Using the explicit formulas for the local Fourier transform [12], we now make this more explicit.

$$\begin{aligned} \mathcal{F}^{(\infty, \infty)} \nabla_{\text{KS}} &\cong \mathcal{F}^{(\infty, \infty)} \left( \bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z)) \right) \\ &\cong \bigoplus_{i=0}^{n-1} \mathcal{F}^{(\infty, \infty)} ((\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z))) \end{aligned}$$

Write  $f(z) = \sum_{i \leq r} t_i z^i$  with  $t_r \neq 0$  and write

$$(\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z)) = (\mathbb{C}((\zeta)), \partial_\zeta + \zeta^{-2} \zeta_n^i f(\zeta_n^i \zeta^{-1})) =: (\mathbb{C}((\zeta)), \partial_\zeta + g(\zeta)/\zeta)$$

Hence

$$g = \frac{\zeta_n^i f(\zeta_n^i \zeta^{-1})}{\zeta} = * \frac{1}{\zeta^{1+r}} + \text{higher order terms}$$

It follows for example from the work of Graham-Squire [12] that

$$\mathcal{F}^{(\infty, \infty)}(\mathbb{C}(\zeta), \partial_\zeta + g(\zeta)/\zeta) \cong (\mathbb{C}(\hat{\zeta}), \partial_{\hat{\zeta}} + h(\hat{\zeta})/\hat{\zeta})$$

where  $g$  and  $h$  are related in the following manner:

$$g = \frac{1}{\zeta \hat{\zeta}}$$

$$h = -g + \frac{r+1}{2r}$$

Hence

$$\hat{\zeta} = (\zeta_n^i f(\zeta_n^i \zeta^{-1}))^{-1}$$

Let  $f^{-1}$  be the compositional inverse of  $f$ . Then

$$\zeta_n^{-i} f^{-1}(\zeta_n^i \hat{\zeta}) = \zeta^{-1}$$

Then

$$h = -\frac{\zeta_n^i f(\zeta_n^i \zeta^{-1})}{\zeta} + \frac{r+1}{2r} = -\zeta_n^{2i} \hat{\zeta} \zeta_n^{-i} f^{-1}(\zeta_n^i \hat{\zeta}) + \frac{r+1}{2r}$$

Hence

$$\begin{aligned} \mathcal{F}^{(\infty, \infty)}(\mathbb{C}(\zeta), \partial_\zeta + \zeta_n^i f(\zeta_n^i \zeta)) &\cong (\mathbb{C}(\hat{\zeta}), \partial_{\hat{\zeta}} + h(\hat{\zeta})/\hat{\zeta}) \\ &\cong (\mathbb{C}(\hat{\zeta}), \partial_{\hat{\zeta}} + \frac{1+r}{2r\hat{\zeta}} - \zeta_n^i f^{-1}(\zeta_n^i \hat{\zeta})) \\ &\cong (\mathbb{C}(\zeta), \partial_\zeta - \frac{1+r}{2r\zeta} + \zeta_n^i f^{-1}(\zeta_n^i \zeta)) \end{aligned}$$

One can make this result even more explicit by giving explicit formulas for the coefficients of the compositional inverse function of  $f$ .

To conclude this section we now make explicit for the UMM the relation between the potential function of the matrix model and the gauge equivalence class of the the KS connection.

**Corollary 1.** *The KS connection  $\nabla_{\text{KS}}$  of the symmetric unitary matrix model with potential  $V(X) = \sum_{i \geq 0} t_i X^i$  satisfies*

$$\nabla_{\text{KS}} \cong \left( \mathbb{C}(\zeta), \partial_\zeta + \sum -(2i+1)t_{2i+1}\zeta^{2i} \right) \oplus \left( \mathbb{C}(\zeta), \partial_\zeta - \sum -(2i+1)t_{2i+1}\zeta^{2i} \right)$$

*Proof.* It is shown in [2] that the coefficients  $a_i$  of the Kac-Schwarz operator are related to the deformation parameters  $t_i$  from the  $k$ 'th multicritical point via  $a_i = 0$  for odd  $i$  and  $t_i = 0$  for even  $i$  and

$$a_{2i} = -(2i+1)t_{2i+1}$$

Hence the result follows from Theorem 1.  $\square$

### 3.1.2 Classical limit

As can be seen in the work of Dijkgraaf-Hollands-Sulkovski [7], one should expect a relation between a suitably defined classical limit of the D-module describing the quantum field theory partition function and a suitable spectral curve. In this section we give a definition of this classical limit for the KS connections. It turns out that in this manner,

in the special case of the UMM quiver, one essentially recovers the spectral curve of the matrix model.

To define the classical limit of the quiver or the string equation one should really consider the variant of the string equation  $[\tilde{P}, \tilde{Q}] = 1$  of the form  $[\tilde{P}, \tilde{Q}] = \hbar$  where  $\hbar$  is an indeterminate. To do so, we define  $\hbar$ -dependent versions of the previously defined quivers. We focus on the following case:

**Definition 8.** For  $f \in \mathbb{C}[z, z^{-1}]$  the  $\hbar$ -dependent string quiver of degree  $n$  is a  $z$ -cyclic quiver  $(V_1, \dots, V_n)$  such that

$$A_{\hbar} V_i := (\hbar \partial_z + f(z)) V_i \subseteq V_{\sigma(i)}$$

such that the permutation  $\sigma$  is the inverse permutation of the one coming from the  $z$ -action:

$$\sigma = (n \ n-1 \ \dots \ 1)$$

We denote this quiver by  $\text{Quiv}_f(\sigma, \hbar)$ .

Instead of connections on the punctured disc one then obtains  $\hbar$ -connections and hence objects interpolating between Higgs bundles and connections, in the sense of non-abelian Hodge theory: In the limit  $\hbar \rightarrow 0$  one obtains Higgs bundles and for  $\hbar = 1$  such an object is a usual connection. Namely:

**Definition 9.** An  $\hbar$ -connection on  $\text{Spec } \mathbb{C}((t))$  is a finite-dimensional  $\mathbb{C}((t))$ -vector space  $V$  with  $\mathbb{C}$ -linear endomorphism  $\nabla$  of  $V$  such that

$$\nabla(fv) = \hbar(\partial_t f)v + f\nabla(v)$$

for all  $f \in \mathbb{C}((t))$  and all  $v \in V$ .

For  $\hbar$ -connections the classical limit is well defined:

**Lemma 3.2.** *The classical limit of an  $\hbar$ -connection  $\hbar\partial + M$  is the algebraic curve defined by the vanishing of the characteristic polynomial:*

$$\text{char}(M|_{\hbar=0}, y) = 0$$

where  $y$  is an indeterminate. This curve is well defined on the gauge equivalence class of the connection.

*Proof.* Let  $n$  be the dimension of the connection. Under the gauge transformation corresponding to  $g \in \mathfrak{gl}_n \mathbb{C}((t))$  the matrix  $M$  changes by

$$M \mapsto g^{-1} M g + \hbar g^{-1} \partial_t(g)$$

Hence in the  $\hbar \rightarrow 0$  limit, the characteristic polynomial of the matrix describing the  $\hbar$ -connection is well defined on the gauge equivalence class.  $\square$

As a corollary of Theorem 1 one can show:

**Corollary 2.** *The classical limit of the KS connection of the degree  $n$  string quiver  $\text{Quiv}_f(\sigma, \hbar)$  is given by the algebraic curve*

$$\prod_{i=0}^{n-1} (y - \zeta_n^i f(\zeta_n^i z)) = 0$$

where  $\zeta_n$  is a primitive  $n$ 'th root of unity.

*Proof.* When the calculations of the proof of Theorem 1 are redone incorporating the above defined  $\hbar$ -dependence one sees that there is a basis in which the  $\hbar$ -connection is of the form

$$\hbar \partial_z + \begin{bmatrix} f(z) & & & \\ & \zeta_n f(\zeta_n z) & & \\ & & \ddots & \\ & & & \zeta_n^{n-1} f(\zeta_n^{n-1} z) \end{bmatrix}$$

and the corollary follows.  $\square$

For the UMM quiver the classical limit can be related to the spectral curve of the matrix model:

For example if the potential function is  $V(X) = X$ , the spectral curve is described by Dijkgraaf-Vafa in [9] as  $y^2 + \cos x + u = 0$ . In a suitable normalization this matches with the classical limit of the corresponding KS connection. This is in accordance with the general philosophy of the results of Dijkgraaf-Hollands-Sulkowski [7] where D-modules are obtained from suitable quantization of spectral curves.

### 3.3 Companion matrix connection

We now define a second type of D-module attached to quivers, the so-called companion matrix connection. This can be defined for permutation quivers, and hence in greater generality than the KS connections which were defined only for string quivers. Throughout this section consider a degree  $n$  permutation quiver  $\text{Quiv}_f(\sigma)$ .

Let

$$\pi : \mathbb{C}((1/z)) \rightarrow \mathbb{C}[z]$$

be the projection map to the polynomial part of a Laurent series with respect to  $1/z$ . Recall that for a point  $V$  in the big cell of the Sato Grassmannian there is a unique element in the intersection of  $V$  and  $\pi^{-1}(1)$ , we denote it by  $\pi^{-1}(1)_V$ . Furthermore, we define:

**Definition 10.** Let  $s \in \mathbb{Z}^{\geq 0}$ . For a permutation  $\sigma \in S_n$  define the subset  $\mathcal{B}(\sigma, s)$  of  $\mathfrak{gl}_n(\mathbb{C}[z])$  to consist of matrices  $B = (B_{ij})_{i,j}$  satisfying  $B_{ij} = \sum_k b_{ijk} z^k$  with

- (i)  $b_{ijk} = 0$  unless possibly for  $k \equiv \sigma(i) - j \pmod{n}$ , where coefficients are interpreted modulo  $n$
- (ii) the degree of  $B$  with respect to  $z$  equals  $s$  and  $b_{ijs} = 0$  unless  $\sigma(i) - j \equiv s \pmod{n}$ . For all such  $i, j$  the value  $b_{ijs}$  is constant.

So for example if  $n = 2$  and  $\sigma = (1\ 2)$  and  $s = 2k$  is even, then  $\mathcal{B}(\sigma, 2k)$  consists of matrices in  $\mathfrak{gl}_2(\mathbb{C}[z])$  with odd polynomials of degree less than  $2k$  on the diagonal and even polynomials of degree  $2k$  on the off-diagonal entries. This is precisely the relevant case for the unitary matrix model, as was described already in [2].

Without the  $A$  constraints, it is clear how to describe the solutions  $(V_1, \dots, V_n)$  to the permutation quiver  $\text{Quiv}_f(\sigma)$ : For  $1 \leq i \leq n$  let  $\phi_i = \pi^{-1}(1)_{V_i}$ . Then

$$V_i = \text{span}_{\mathbb{C}}(\phi_i, z\phi_{i-1}, z^2\phi_{i-2}, \dots, z^n\phi_i, \dots)$$

where the indices are considered modulo  $n$ . There are no further constraints, one can choose the  $\phi_i$ 's arbitrarily in order to solve for the quiver. Now we add the  $A$  constraint:

They yield that for all  $1 \leq i \leq n$  one has

$$A\phi_i = (\partial_z + f(z))\phi_i = \sum_{1 \leq j \leq n} B_{ij}\phi_j \quad \text{for a unique } B \in \mathcal{B}(\sigma, \deg f)$$

It follows that up to an exponential factor the  $\phi_i$ 's yield a flat section of the connection  $\partial_z - B$ :

$$(\partial_z - B) \exp \left( \int f(z) dz \right) \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} = \bar{0}$$

**Definition 11.** The companion matrix connection of a degree  $n$  permutation quiver is defined as

$$\nabla_{\text{Comp}} = (\mathbb{C}((z))^n, \partial_z - B)$$

where  $B \in \mathcal{B}(\sigma, \deg f)$  as above

This definition makes sense for arbitrary permutation quivers, one need not assume that the quiver is a string quiver.

We now describe the gauge equivalence class of the companion matrix connection.

**Theorem 2.** Let  $f \in \mathbb{C}[z]$  with leading order term  $\alpha z^m$  and consider an associated permutation quiver  $\text{Quiv}_f(\sigma)$  of degree  $n$ , where  $\sigma$  is an  $n$ -cycle. The companion matrix connection satisfies

$$\nabla_{\text{Comp}} \cong \bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \lambda_i)$$

such that:

(i) For all  $0 \leq i \leq n-1$  one has

$$\lambda_i = \zeta_n^i \alpha z^m + \text{lower order terms}$$

where  $\zeta_n$  is a primitive  $n$ 'th root of unity.

(ii)  $\lambda_0 = f$ .

*Proof.* Let  $\sigma$  denote the permutation associated to the permutation quiver. As explained before, the matrix  $B$  in the definition of the companion matrix is in  $\mathcal{B}(\sigma, \deg f)$ . In particular, the coefficient matrix of the leading  $z$  power in  $B$  is a permutation matrix with eigenvalues  $1, \zeta_n, \dots, \zeta_n^{n-1}$ , where  $\zeta_n$  is a primitive  $n$ 'th root of unity, since we assume that  $\sigma$  is an  $n$ -cycle. It follows from the Levelt-Turrittin algorithm that  $\nabla_{\text{Comp}}$  is the direct sum of 1-dimensional connections.

Furthermore, from the definitions it is clear that  $f$  is an exponential factor of  $\nabla_{\text{Comp}}$  and hence  $(\mathbb{C}((z)), \partial_z + f)$  is a direct summand of  $\nabla_{\text{Comp}}$ . Again from the Levelt-Turrittin algorithm one can deduce that the leading order terms of all the  $n$  exponential factors of the connection are of the form  $\zeta_n^i \alpha z^m$  for  $1 \leq i \leq n$ , as desired.  $\square$

Note that in general  $\nabla_{\text{Comp}}$  is not isomorphic to  $\bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z))$ , in contrast to the KS connection. For example, if  $\sigma = (1 \ 4 \ 2 \ 5 \ 3) \in S_5$  then the exponents of

powers of  $z$  occurring in entries of  $B$  have to satisfy the congruence modulo 5 summarized by

$$\begin{bmatrix} 3 & 2 & 1 & 5 & 4 \\ 4 & 3 & 2 & 1 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 5 & 4 & 3 & 2 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}$$

Hence, one can take for example

$$\nabla_{\text{Comp}} = (\mathbb{C}((z))^5, \partial_z - \begin{bmatrix} -3z^3 & -6z^2 & 0z & 1z^5 & -5z^4 \\ 2z^4 & -2z^3 & 2z^2 & 2z & 1z^5 \\ 1z^5 & -3z^4 & -4z^3 & -3z^2 & -1z \\ -3z & 1z^5 & -3z^4 & 3z^3 & 1z^2 \\ -5z^2 & 0z & 1z^5 & -5z^4 & 4z^3 \end{bmatrix})$$

Let

$$f(z) = \frac{26836}{625}z + \frac{1599}{125}z^2 - \frac{2}{5}z^3 - \frac{14}{5}z^4 + z^5$$

Using the Levelt-Turrittin algorithm one sees that

$$\nabla_{\text{Comp}} \cong \bigoplus_{i=0}^4 (\mathbb{C}((z)), \partial_z + \zeta_5^{2i} f(\zeta_5^i z))$$

where  $\zeta_5$  is a primitive 5'th root of unity. This is a special case of the following calculation that gives the Levelt-Turrittin normal form of  $\nabla_{\text{Comp}}$  whenever the permutation  $\sigma$  is such that there is  $k$  such that

$$\sigma(i) - i \equiv k \pmod{n}$$

We claim that under this assumption the companion matrix connection of  $\text{Quiv}_f(\sigma)$  satisfies

$$\nabla_{\text{Comp}} \cong \bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \zeta_n^{-ki} f(\zeta_n^i z))$$

where  $\zeta_n$  is a primitive  $n$ 'th root of unity. So in the above example  $k = 3$ . We now prove the claim:

For any such  $\sigma$  write  $\nabla_{\text{Comp}} \cong (\mathbb{C}((z))^n, \partial_z - M)$  with  $M \in \mathfrak{gl}_n \mathbb{C}((z))$ . As before, one knows that the connection is diagonalizable. Furthermore, the Levelt-Turrittin algorithm show that in fact the corresponding diagonal connection matrix agrees with the diagonal matrix of eigenvalues of  $M$  up to terms of order  $z^{-1}$  or lower. We also know from the previous theorem that one of the diagonal entries of the connection is  $f$ . Let  $\tilde{f}$  be an eigenvalue of  $M$  agreeing with  $f$  up to negative powers of  $z$ . It suffices to show that  $\zeta_n^{-k} f(\zeta_n z)$  is an eigenvalue of  $M$ . Hence it suffices to show that  $M(\zeta_n z)$  is conjugate to  $\zeta_n^k M(z)$ . This is indeed the case: For  $s_i = \sigma(i) - k$  one has

$$\text{diag}(\zeta_n^{s_1}, \dots, \zeta_n^{s_n}) M(\zeta_n^k z) \text{diag}(\zeta_n^{-s_1}, \dots, \zeta_n^{-s_n}) = (\zeta_n^{\sigma(i)-j+s_j-s_i} M(z)_{i,j})_{i,j} = \zeta_n^k M(z)$$

as desired. This proves the claim. As a special case one obtains:

**Theorem 3.** *Fix a degree  $n$  string quiver  $\text{Quiv}_f(\sigma)$ . Then the associated KS connection and companion matrix connection are isomorphic:*

$$\nabla_{\text{KS}} \cong \bigoplus_{i=0}^{n-1} (\mathbb{C}((z)), \partial_z + \zeta_n^i f(\zeta_n^i z)) \cong \nabla_{\text{Comp}}$$

where  $\zeta_n$  is a primitive  $n$ 'th root of unity.

*Proof.* It was shown in Theorem 1 that  $\nabla_{\text{KS}}$  is of the claimed form. Note that the permutation  $\sigma$  in the definition of a string quiver satisfies

$$\sigma(i) - i \equiv -1 \pmod{n}$$

Therefore, the calculation preceding the corollary imply that

$$\bigoplus_{i=0}^{n-1} \left( \mathbb{C}((z)), \partial_z + \zeta_n^{-(n-1)i} f(\zeta_n^i z) \right) \cong \nabla_{\text{Comp}}$$

as desired.  $\square$

Note that Theorem 2 is already sufficient to generalize to much more general quivers the Anagnostopoulos-Bowick-Schwarz description [2] (Section 5) of the moduli space of solutions to the symmetric unitary matrix model:

**Theorem 4.** *The moduli space of solutions to a degree  $n$  permutation quiver  $\text{Quiv}_f(\sigma)$ , with varying  $f$ , is an  $n$ -fold unbranched cover of the space of matrices*

$$\mathcal{B}(\sigma) := \bigcup_{s \geq 0} \mathcal{B}(\sigma, s)$$

*Proof.* Fix for now  $f \in \mathbb{C}[z]$  and consider  $\text{Quiv}_f(\sigma)$ . Since  $B \in \mathcal{B}(\sigma, \deg f)$ , one sees as in the above proof that the companion matrix  $\nabla_{\text{Comp}}$  has  $n$  distinct exponential factors. It follows that there is a unique flat section of the connection with leading order approximated by  $\exp(\int f(z) dz)$ . Furthermore, after diagonalizing the leading order coefficient matrix it is known that the connection can be diagonalized by a gauge transformation of the form

$$g = \text{Id} + \text{lower order terms}$$

and hence the flat section has the desired asymptotics, meaning the Laurent series all start with 1. For the other  $n - 1$  exponential factors of the connection the 1's are replaced by the suitable powers of an  $n$ 'th root of unity. In this manner the collection of all solutions to degree  $n$  quivers is obtained as the  $n$ -fold cover of  $\mathcal{B}(\sigma)$  as desired.  $\square$

## 4 Virasoro constraints

In this section we describe Virasoro constraints for the quivers that we have considered. Suppose  $(V_1, \dots, V_n)$  is a  $z$ -cyclic quiver. Then each point satisfies

$$z^n V_i \subseteq V_i$$

This is known to have the following consequence for the  $\tau$ -function of the KP hierarchy associated to the point  $V_i$ : It is an  $n$ -reduced  $\tau$ -function, hence for example for  $n = 2$  one obtains KdV  $\tau$ -functions for each vertex of the quiver. The situation becomes more interesting in the presence of a Kac-Schwarz operator, hence for the permutation or string quivers. For simplicity of exposition we focus in this section on the latter type of quiver.

We first briefly recall the relation between Virasoro constraints on level of  $\tau$ -functions and in terms of the Sato Grassmannian, as described for example in [11] and [14]. For  $n \in \mathbb{Z}$  define operators on the space  $\mathbb{C}[[t_1, t_2, \dots]]$  by

$$L_n = \frac{1}{2} \sum_{k+l=-n} k l t_k t_l + \sum_{k-l=-n} k t_k \partial_l + \frac{1}{2} \sum_{k+l=n} \partial_k \partial_l$$

and define

$$J_n = \begin{cases} t_n & \text{if } n > 0 \\ -n \partial_{t_{-n}} & \text{if } n < 0 \end{cases}$$

The operators  $L_n$  yield a representation of the Virasoro algebra. The annihilation of KP  $\tau$ -functions by the operators  $L_n$  can be related to symmetry properties of the relevant point of the Sato Grassmannian: For a point  $V$  of  $Gr$  let  $\tau_V$  denote the corresponding KP  $\tau$ -function. It is shown for example in [11] that the following holds:

- (i)  $-z^n (z \frac{d}{dz} + \frac{1+n}{2}) V \subseteq V$  if and only if  $L_n \tau_V = \lambda \tau_V$  for some  $\lambda \in \mathbb{C}$ .
- (ii)  $z^n V \subseteq V$  (say for  $n \geq 0$ ) if and only if  $J_{-n} \tau_V = \mu \tau_V$  for some  $\mu \in \mathbb{C}$ .

In many situations one expects the constants  $\lambda, \mu$  to be 0. The standard way to obtain such a result is to exhibit the relevant operator as a suitable commutator. For example, in the case of the  $(p, q)$  models of 2D gravity one has

$$z^{pi} A^{p,q} = \left[ A^{p,q}, \frac{1}{i+1} z^{p(i+1)} A^{p,q} \right]$$

where  $A^{p,q}$  is the Kac-Schwarz operator as defined in Section 2. This can be used to show that the  $\tau$ -function of the  $(p, q)$  model satisfies

$$\left( L_{p(i-1)} - \frac{p}{2} J_{p(i-1)} + J_{pi+q} \right) \tau(t_1, t_2, \dots) = 0$$

for all  $i \geq 0$ . In the UMM case the situation is more subtle,  $L_0$  is not a commutator. We revisit this aspect in the generality of the string quiver.

One can write down Virasoro constraints for the  $\tau$ -functions of permutation quiver, we will focus for simplicity on the special case of string quivers. First we make the following definition. Fix  $f = \sum r_i z^i \in \mathbb{C}[z]$  and consider a degree  $n$  string quiver. It is useful to define the deformed versions  $L'_j$  of  $L_j$ , depending on the choice of  $f$ . Namely, let

$$L'_j = L_j + \sum_i r_i J_{jn+i+1}$$

**Theorem 5.** *Let  $\tau_1, \dots, \tau_n$  be the KP  $\tau$ -functions associated to a solution to a degree  $n$  string quiver  $\text{Quiv}_f(\sigma)$ . Then:*

- (i) *for all  $1 \leq i \leq n$  and all  $j \geq 1$  one has  $L'_{nj} \tau_i = 0$*
- (ii) *there are constants  $\lambda_1, \dots, \lambda_n$  such that  $L'_0 \tau_i = \lambda_i \tau_i$*

*Proof.* Suppose  $(V_1, \dots, V_n)$  is a solution to this quiver. Then each  $V_i$  satisfies for all  $k \geq 0$

$$z^{kn+1} A V_i \subseteq V_i$$



It follows that there are constants  $\lambda_{k,i}$  such that  $L'_{nk}\tau_{V_i} = \lambda_{k,i}\tau_{V_i}$ . Since for  $k \geq 1$  one has  $L_{nk} = [L_{nk}, L_0]/nk$  it follows that

$$L'_{nk}\tau_i(t_1, t_2, \dots) \equiv 0$$

for all  $k \geq 1$ . For the  $L_0$  action one only knows a priori that  $\tau_i$  is an eigenvector.  $\square$

The phenomenon that the  $L_0$  action on  $\tau$ -functions is more subtle than for  $L_j$  with  $j \geq 1$  occurs already in the UMM quiver and it is natural to compare the  $\lambda_i$ 's. In [13] it was claimed that in the very special case of the UMM, one has  $\lambda_1 = \lambda_2$ . We outline some problems with their arguments:

Their set-up corresponds to the UMM quiver. As described before, starting with a solution  $v$  of the mKdV equation one obtains two related KdV  $\tau$ -functions  $\tau_0, \tau_1$  via the Miura transforms. The  $L_0$  operator of the Virasoro algebra is normalized in [13] so that

$$2L_0 = \sum_{k=0}^{\infty} (2k+1)t_k \partial_{t_k} + \frac{1}{8}$$

Then, as before, it is known that  $L_0\tau_0 = \lambda_1\tau_0$  and  $L_0\tau_1 = \lambda_2\tau_1$  for some constants  $\lambda_1, \lambda_2$  and it is claimed by Hollowood - Miramontes - Nappi - Pasquinucci in [13] that  $\lambda_1 = \lambda_2$ . The function  $v$  is related to  $\tau_0$  and  $\tau_1$  via

$$v = \partial_x \log(\tau_1) - \partial_x \log(\tau_0)$$

The mKdV flow equations are for all  $k \geq 0$  given by

$$\partial_{t_k} v = -\frac{1}{2} \partial_x (\partial_x - 2v) R_k$$

where the  $R_k$ 's are the relevant Gelfand-Dickey polynomials and  $x = t_0$ . The mKdV string equation is given in [13] by

$$\sum_{k=0}^{\infty} (2k+1)t_k (\partial_x - 2v) R_k = 0$$

In combination, by taking the derivative with respect to  $x$ , one obtains since  $R_0 = 1$  that

$$v + \sum_{k=0}^{\infty} (2k+1)t_k \partial_{t_k} (\partial_x \log(\tau_1) - \partial_x \log(\tau_0)) = v + \sum_{k=0}^{\infty} (2k+1)t_k \partial_{t_k} v = 0$$

Hence

$$\partial_x \left[ \sum_{k=0}^{\infty} (2k+1)t_k \left( \frac{1}{\tau_1} \partial_{t_k} \tau_1 - \frac{1}{\tau_0} \partial_{t_k} \tau_0 \right) \right] = \partial_x \left[ \sum_{k=0}^{\infty} (2k+1)t_k \partial_{t_k} (\log(\tau_1) - \log(\tau_0)) \right] = 0$$

and therefore

$$\frac{1}{\tau_1} (2L_0\tau_1 - \frac{\tau_1}{8}) = \frac{1}{\tau_0} (2L_0\tau_0 - \frac{\tau_0}{8}) + \text{constant}$$

Hence

$$2\lambda_2 - \frac{1}{8} = 2\lambda_1 - \frac{1}{8} + \text{constant}$$

The authors of [13] then implicitly assume the constant on the right hand side is 0

and deduce that  $\lambda_1 = \lambda_2$ . In the absence of further results concerning the integration constants, the argument is not complete. Note also that scaling dimension arguments do not in an obvious manner determine the part of the integration constant that is an absolute constant, meaning independent of all the time variables  $t_k$ .

It remains an interesting problem to understand the relation between  $\lambda_1$  and  $\lambda_2$  and more generally between the  $n$  eigenvalues of the  $L_0$  operator acting on the  $n$  KP  $\tau$ -functions associated to a string quiver.

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